

Improper Graceful and Odd-graceful Labellings of Graph Theory

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Abstract

In this paper we define some new labellings for trees, called the in-improper and out-improper odd-graceful labellings such that some trees labelled with the new labellings can induce graceful graphs having at least a cycle. We, next, apply the new labellings to construct large scale of graphs having improper graceful/odd-graceful labellings or having graceful/odd-graceful labellings.

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1 Introduction and concepts

Graph colorings/labellings of graph theory can be applied in many areas of science such as bioinformatics, (scale-free, small-world) networks, VLSI. As known, colored/labelled graphs are used in many areas of today's science and computer science. For example, some design algorithms are related with such graphs; the problem of frequency distribution of communication can be investigated by graph distinguishing colorings. We try to explore connection between labelled graphs and cryptographs of information security for graphic cryptograph. The survey article [2] in which more than 1400 papers on various graph labellings were collected provides a good overview of concepts and results of current graph labellings. It has been known that all lobsters are odd-graceful (Ref. [5]), however “*all lobsters are graceful*” conjectured by Bermond [1] is open up now. In [11], the author investigated the so-called *relaxed graceful labelling* of trees. We are motivated from the relaxed graceful labelling, and then propose some improper graceful labellings for trees in order to producing graceful graphs having at least a cycle.

We use standard terminology and notation of graph theory, and all graphs mentioned here are simple, undirected and finite. For the sake of simplicity, the shorthand symbol $[m, n]$ stands for a set $\{m, m+1, \dots, n\}$, where m and n are integers with $0 \leq m < n$; the notation $[s, t]^o$ indicates an odd-set $\{s, s+2, \dots, t\}$, where s and t both are odd integers with $1 \leq s < t$. A *leaf* is a vertex of degree one, and $L(G)$ stands for the set of leaves of a graph G . A *caterpillar* is a tree T such that the deletion of leaves of T results in a path. A *lobster* is a tree such that the graph obtained by removing all leaves of the tree is just a caterpillar. A (p, q) -graph is a graph having p vertices and q edges.

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Definition 1. Suppose that a connected (p, q) -graph G admits a mapping $\theta : V(G) \rightarrow \{0, 1, 2, \dots\}$. For every edge $xy \in E(G)$ the induced edge label is defined as $\theta(xy) = |\theta(x) - \theta(y)|$, and write $\theta(V(G)) = \{\theta(u) : u \in V(G)\}$, $\theta(E(G)) = \{\theta(xy) : xy \in E(G)\}$. There are the following constraints:

- (a) $|\theta(V(G))| = p$.
- (b) $\theta(V(G)) \subseteq [0, q]$, $\min \theta(V(G)) = 0$.
- (c) $\theta(V(G)) \subset [0, 2q - 1]$, $\min \theta(V(G)) = 0$.
- (d) $\theta(E(G)) = [1, q]$.
- (e) $\theta(E(G)) = [1, 2q - 1]^o$.
- (f) G is a bipartite graph with the bipartition (X, Y) such that $\max\{\theta(x) : x \in X\} < \min\{\theta(y) : y \in Y\}$ ($\theta(X) < \theta(Y)$ for short).
- (g) G is a tree containing a perfect matching M such that $\theta(x) + \theta(y) = q$ for each edge $xy \in M$.
- (h) G is a tree having a perfect matching M such that $\theta(x) + \theta(y) = 2q - 1$ for each edge $xy \in M$.

Then θ is a *graceful labelling* if it holds (a), (b) and (d); θ is a *set-ordered graceful labelling* if it holds (a), (b), (d) and (f); θ is a *strongly graceful labelling* if it holds (a), (b), (d) and (g); θ is a *strongly set-ordered graceful labelling* if it holds (a), (b), (d), (f) and (g); θ is an *odd-graceful labelling* if it holds (a), (c) and (e); θ is a *set-ordered odd-graceful labelling* if it holds (a), (c), (e) and (f); θ is a *strongly odd-graceful labelling* if it holds (a), (c), (e) and (h); θ is a *strongly set-ordered odd-graceful labelling* if it holds (a), (c), (e), (f) and (h).

Also, graceful/odd-graceful labellings are called *proper graceful/odd-graceful labellings* in the following discussion.

Definition 2. Let G be a (p, q) -graph, and f be a mapping from $V(G)$ to $\{0, 1, 2, \dots\}$ such that $\min f(V(G)) = 0$ and $\max f(V(G)) = m$, where $f(V(G)) = \{f(x) : x \in V(G)\}$. Write $f(E(G)) = \{f(uv) = |f(u) - f(v)| : uv \in E(G)\}$. There are four restrictions:

- C1. $f(E(G)) = [1, q]$ and $m \leq q$; C2. $f(E(G)) = [1, q]$ and $m > q$;
- C3. $f(E(G)) = [1, 2q - 1]^o$ and $m \leq 2q - 1$; C4. $f(E(G)) = [1, 2q - 1]^o$ and $m > 2q - 1$.

Then, f is called an *in-improper graceful labelling* (in-imagl) if f holds C1, an *out-improper graceful labelling* (out-imagl) if f holds C2, an *in-improper odd-graceful labelling* (in-imoddgl) if f holds C3, and an *out-improper odd-graceful labelling* (out-imoddgl) if f holds C4.

The result in Theorem 1 have been shown in some literature (Ref. [2]). We show a short proof of Theorem 1 here.

Theorem 1. *Every tree admits (i) in-imagls or out-imagls; and (ii) in-imoddgls or out-imoddgls.*

Proof. We present the proof of the assertion (i) by introduction on orders of trees, since the proof of the assertion (ii) is very similar with that of the assertion (i).

Let T be a tree on n vertices, so T has $(n - 1)$ edges. For $n = 2$, T admits a graceful labelling. Suppose that the theorem holds true for trees having smaller numbers of vertices. We consider a tree $H = T - v$, where v is a leaf of T and is adjacent to u in T . So H has an in-imgl or an out-imgl f such that $f(E(H)) = [1, n - 2]$ by induction hypothesis. We label the vertex v of T as $n - 1 + f(u)$. Thereby, we obtain an out-imgl g as: $g(v) = n - 1 + f(u)$, $g(x) = f(x)$ for $x \in V(H) = V(T) \setminus \{v\}$. Clearly, $g(E(T)) = [1, n - 1]$. The proof of the theorem is complete. \square

Based on Theorem 1 there are two parameters under a labelling f of a tree T on n vertices: (1) $k = |S|$, where S is a maximum subset of $V(T)$ such that every vertex $u \in S$ holds $f(u) = f(u')$ for some $u' \in S \setminus \{u\}$; (2) $l = n - c$, where c is the number of distinct labels assigned to the vertices of T . We call f an $in-(k, l)$ -imgl (resp. $out-(k, l)$ -imgl) as if f is an in-imgl (resp. out-imgl) of T , and furthermore we call f a *proper graceful labelling* if $(k, l) = (0, 0)$; and we call f an $in-(k, l)$ -imoddgl (resp. $out-(k, l)$ -imoddgl) if f is an in-imoddgl (resp. out-imoddgl) of T , and furthermore we call f a *proper odd-graceful labelling* when $(k, l) = (0, 0)$. There exist the following possible parameters:

$$\begin{aligned} l_{in}^{\max} &= \max\{l : in-(k, l)\text{-imgls}\} \text{ and } k_{in}^{\max} = \max\{k : in-(k, l_{in}^{\max})\text{-imgls}\}; \\ l_{in}^{\min} &= \max\{l : in-(k, l)\text{-imgls}\} \text{ and } k_{in}^{\min} = \max\{k : in-(k, l_{in}^{\min})\text{-imgls}\}; \\ l_{out}^{\min} &= \min\{l : out-(k, l)\text{-imgls}\} \text{ and } k_{out}^{\min} = \min\{l : out-(k, l_{out}^{\min})\text{-imgls}\}. \end{aligned}$$

Similarly, it may be interesting to consider: *Determine $in-(k_{in}^{\max}, l_{in}^{\max})$ -imoddgls and $out-(k_{in}^{\min}, l_{in}^{\min})$ -imoddgls of trees*, where $l_{in}^{\max} = \max\{l : in-(k, l)\text{-imoddgls}\}$, $k_{in}^{\max} = \max\{k : in-(k, l_{in}^{\max})\text{-imoddgls}\}$, $l_{in}^{\min} = \min\{l : in-(k, l)\text{-imoddgls}\}$ and $k_{in}^{\min} = \min\{k : in-(k, l_{in}^{\min})\text{-imoddgls}\}$.

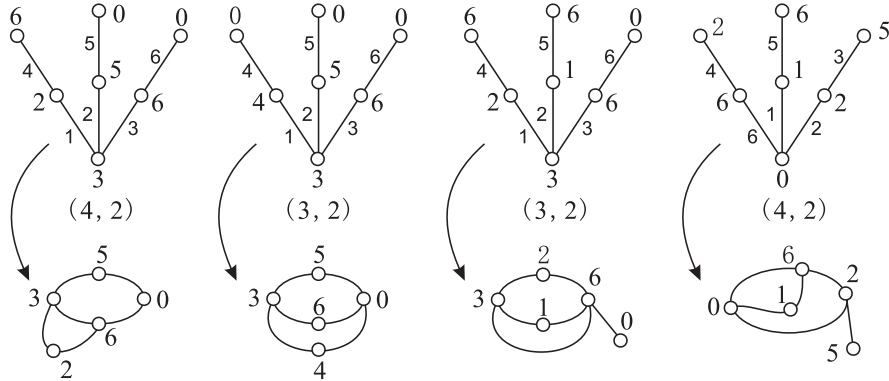


Figure 1: A lobster having four in-improper graceful labellings induces four graceful graphs.

Rosa [4] in 1966 conjectured: *Every tree is graceful*. Thereby, Rosa's conjecture implies: Every tree admits in-improper graceful labellings, and furthermore every $in-(k_{in}^{\min}, l_{in}^{\min})$ -imgl is equal to a graceful labelling in a tree. In 1991 Gnanaiothi [3] gave a conjecture: *Every tree is odd-graceful*, which leads that every tree admits in-improper odd-graceful labellings; and furthermore each $in-(k_{in}^{\min}, l_{in}^{\min})$ -imoddgl is an odd-graceful labelling in a tree.

2 Finding in-imgl and in-imoddgl of graphs

The term “adding leaves to a (p, q) -graph G ” means that we join new vertices $u_{i,1}, u_{i,2}, \dots, u_{i,m_i}$ to each vertex u_i of the (p, q) -graph G for $i \in [1, p]$, here, some m_i may be zero.

Theorem 2. *Suppose that a tree T admit set-ordered graceful/odd-graceful labellings. Then adding leaves to T produces a tree that admits in-imgl and proper odd-graceful labellings.*

Proof. A tree T has its own bipartition $V(T) = X \cup Y$ with $X = \{x_1, x_2, \dots, x_s\}$ and $Y = \{y_1, y_2, \dots, y_t\}$ such that $s + t = |T|$, $u \in X$ and $v \in Y$ for each edge $uv \in E(T)$. Adding leaves to T produces a tree H . So, we have that every vertex x_i of the tree H is added the leaves $x_{i,1}, x_{i,2}, \dots, x_{i,m_i}$ for $i \in [1, s]$, and every vertex y_j of the tree H is added the leaves $y_{j,1}, y_{j,2}, \dots, y_{j,n_j}$ for $j \in [1, t]$. Here, it may be that some $m_i = 0$ or $n_j = 0$. Let $M_x = \sum_{k=1}^s m_k$ and $M_y = \sum_{k=1}^t n_k$.

First, we consider the case: T has a set-ordered graceful labelling f such that $0 = f(x_1) \leq f(x_i) < f(x_{i+1})$ for $i \in [1, s-1]$, $f(x_s) < f(y_1)$, and $f(y_i) < f(y_{i+1}) \leq f(y_t) = s + t - 1$ for $i \in [1, t-1]$. We define a labelling g of the tree H as follows.

Step 1. We label the edges of H in the following way: $g(x_1x_{1,a}) = a$ for $a \in [1, m_1]$, $g(x_ix_{i,a}) = a + \sum_{k=1}^{i-1} m_k$ for $a \in [1, m_i]$ and $i \in [2, s]$; $g(y_t y_{t,b}) = M_x + b$ for $b \in [1, n_t]$, $g(y_{t-j+1} y_{t-j+1,b}) = M_x + b + \sum_{k=1}^{j-1} n_{t-k+1}$ for $b \in [1, n_j]$ and $j \in [2, t]$; $g(y_1 y_{1,b}) = M_x + M_y - n_1 + b$ for $b \in [1, n_1]$; $g(x_i y_j) = f(x_i y_j) + M_x + M_y$ for $x_i y_j \in E(T)$.

Step 2. We label the vertices of the tree H by setting $g(x_i) = f(x_i)$ for $i \in [1, s]$; $g(y_j) = f(y_j) + M_x + M_y$ for $j \in [1, t]$; $g(x_{i,a}) = g(x_i) + g(x_i x_{i,a})$ for $a \in [1, m_i]$ and $i \in [1, s]$; $g(y_{j,b}) = g(y_j) - g(y_j y_{j,b})$ for $b \in [1, n_j]$ and $j \in [1, t]$. Notice that $g(x_i) < g(x_{i,a}) = g(x_i) + g(x_i x_{i,a}) \leq f(x_s) + \sum_{k=1}^i m_k$ for $a \in [1, m_i]$ and $i \in [1, s]$; and for $b \in [1, n_j]$ and $j \in [1, t]$, we then have

$$\begin{aligned} g(y_j) &> g(y_{j,b}) = g(y_j) - g(y_j y_{j,b}) \\ &= f(y_j) + M_x + M_y - \left(M_x + b + \sum_{k=1}^{j-1} n_{t-k+1} \right) \\ &\geq f(y_j) + \sum_{k=1}^{t-j} n_k. \end{aligned}$$

Step 3. Notice that $0 = g(x_1) < g(u) \leq g(y_t) = s + t - 1 + M_x + M_y = |H| - 1$ for $u \in V(H)$, and

$$\begin{aligned} g(E(H)) &= \{g(x_i x_{i,a}) : a \in [1, m_i], i \in [1, s]\} \cup \{g(y_j y_{j,b}) : b \in [1, n_j], j \in [1, t]\} \cup g(E(T)) \\ &= [1, |E(H)| - 1]. \end{aligned}$$

It follows that g is an in-imgl of the tree H .

Second, to prove that the tree H admits an in-imoddgl, we take a set-ordered odd-graceful labelling α of T . By the definition of a set-ordered odd-graceful labelling, we have $0 = \alpha(x_1) \leq$

$\alpha(x_i) < \alpha(x_{i+1})$ for $i \in [1, s-1]$, $\alpha(x_s) < \alpha(y_1)$, and $\alpha(y_i) < \alpha(y_{i+1}) \leq \alpha(y_t) = 2|T| - 3$ for $i \in [1, t-1]$. Since $\alpha(E(T)) = [1, 2|T| - 3]^o$, so the vertices of X have the same parity, so do the vertices of Y . Without loss of generality, we assume that each $\alpha(x_i)$ is even, and each $\alpha(y_j)$ is odd. We define another labelling β of the tree H as follows.

We, now, label the edges of the tree H by defining $\beta(x_1x_{1,a}) = 2a - 1$ for $a \in [1, m_1]$, $\beta(x_ix_{i,a}) = 2a - 1 + 2\sum_{k=1}^{i-1} m_k$ for $a \in [1, m_i]$ and $i \in [2, s]$; $\beta(y_ty_{t,b}) = 2M_x - 1 + 2b$ for $b \in [1, n_t]$, $\beta(y_{t-j+1}y_{t-j+1,b}) = 2M_x - 1 + 2b + 2\sum_{k=1}^{j-1} n_{t-k+1}$ for $b \in [1, n_j]$ and $j \in [2, t]$; $\beta(y_1y_{1,b}) = 2(M_x + M_y - n_1 + b) - 1$ for $b \in [1, n_1]$; $\beta(x_iy_j) = \alpha(x_iy_j) + 2(M_x + M_y)$ for $x_iy_j \in E(T)$.

We, next, label the vertices of the tree H by setting $\beta(x_i) = \alpha(x_i)$ for $i \in [1, s]$; $\beta(y_j) = \alpha(y_j) + 2(M_x + M_y)$ for $j \in [1, t]$; $\beta(x_{i,a}) = \beta(x_i) + \beta(x_ix_{i,a})$ for $a \in [1, m_i]$ and $i \in [1, s]$ (here $\beta(x_{i,a})$ is odd); $\beta(y_{j,b}) = \beta(y_j) - \beta(y_jy_{j,b})$ for $b \in [1, n_j]$ and $j \in [1, t]$ (here $\beta(y_{j,b})$ is even).

Notice that $\beta(x_i) < \beta(x_{i,a}) = \beta(x_i) + \beta(x_ix_{i,a}) \leq \alpha(x_s) - 1 + 2\sum_{k=1}^i m_k$ for $a \in [1, m_i]$ and $i \in [1, s]$. For $b \in [1, n_j]$ and $j \in [1, t]$, we so have

$$\begin{aligned} \beta(y_j) &> \beta(y_{j,b}) = \beta(y_j) - \beta(y_jy_{j,b}) \\ &= \alpha(y_j) + 2(M_x + M_y) - \left(2M_x - 1 + 2b + 2\sum_{k=1}^{j-1} n_{t-k+1} \right) \\ &\geq \alpha(y_j) - 1 + 2\sum_{k=1}^{t-j} n_k. \end{aligned}$$

So, no two vertices u and v of H hold $\beta(u) = \beta(v)$. It follows

$$\begin{aligned} \beta(E(H)) &= \{\beta(x_ix_{i,a}) : a \in [1, m_i], i \in [1, s]\} \cup \{\beta(y_jy_{j,b}) : b \in [1, n_j], j \in [1, t]\} \cup \beta(E(T)) \\ &= [1, 2|E(H)| - 1]^o \end{aligned}$$

that g is a proper odd-graceful labelling of the tree H . The theorem is covered. \square

For a lobster T , we have a caterpillar $T - L(T)$ by the definition of a lobster. Notice that any caterpillar admits set-ordered graceful/odd-graceful labellings [5]. So, the following Corollary 3 is an immediate consequence of Theorem 2.

Corollary 3. *Every lobster admits in-imgl and proper odd-graceful labellings.*

Theorem 4. *Let G be a connected bipartite graph having vertices w_1, w_2, \dots, w_n . For each fixed integer $i \in [1, n]$, identifying a certain vertex of every connected bipartite graph $G_{i,j}$ for $j \in [1, m_i]$ (it is allowed some $m_i = 0$) with w_i into a vertex produces a connected graph H^* . Then H^* admits an in-imgl (resp. in-imoddgl) if G and every $G_{i,j}$ admit set-ordered graceful labellings (resp. set-ordered odd-graceful labellings).*

Proof. We show the proof of the first case “ H^* admits an in-imgl if G and every $G_{i,j}$ admit set-ordered graceful labellings”, and the proof for the second case “ H^* admits an in-imoddgl if G and

every $G_{i,j}$ admit set-ordered odd-graceful labellings" can be omitted since it is the same as that of the first case.

By the theorem's hypothesis, the connected bipartite graph G has its own bipartition $V(G) = X \cup Y$ with $X = \{u_i : i \in [1, s]\}$ and $Y = \{v_j : j \in [1, t]\}$, and has a set-ordered graceful labelling f such that $0 = f(u_1) \leq f(u_i) < f(u_{i+1})$ for $i \in [1, s-1]$, $f(u_s) < f(v_1)$, and $f(v_i) < f(v_{i+1}) \leq f(v_t) = |E(G)|$ for $i \in [1, t-1]$.

For a fixed integer $i \in [1, s]$, we will identify a certain vertex of every connected bipartite $(p_{i,j}^u, q_{i,j}^u)$ -graph $G_{i,j}^u$ with the vertex $u_i \in X$ into a vertex, $j \in [1, m_i]$. Let $V(G_{i,j}^u) = X_{i,j}^u \cup Y_{i,j}^u$ be the bipartition of $V(G_{i,j}^u)$, where $X_{i,j}^u = \{x_{i,j,l}^u : l \in [1, m_{i,j}^u]\}$ and $Y_{i,j}^u = \{y_{i,j,r}^u : r \in [1, n_{i,j}^u]\}$; $G_{i,j}^u$ has a set-ordered graceful labelling $f_{i,j}$ such that $0 = f_{i,j}(x_{i,j,1}^u) \leq f_{i,j}(x_{i,j,l}^u) < f_{i,j}(x_{i,j,l+1}^u)$ for $l \in [1, m_{i,j}^u - 1]$, $f_{i,j}(x_{i,j,m_{i,j}^u}^u) < f_{i,j}(y_{i,j,1}^u)$, $f_{i,j}(y_{i,j,r}^u) < f_{i,j}(y_{i,j,r+1}^u) \leq f_{i,j}(y_{i,j,n_{i,j}^u}^u) = q_{i,j}^u$ for $r \in [1, n_{i,j}^u - 1]$. Let $M_u = \sum_{k=1}^s M_k$, where $M_i = \sum_{k=1}^{m_i} q_{i,k}^u$ for $i \in [1, s]$.

For a fixed integer $i \in [1, t]$, a certain vertex of each connected bipartite $(p_{i,j}^v, q_{i,j}^v)$ -graph $G_{i,j}^v$ is identified with $v_i \in Y$ into a vertex, $j \in [1, n_i]$. Let $V(G_{i,j}^v) = X_{i,j}^v \cup Y_{i,j}^v$ be the bipartition of $V(G_{i,j}^v)$, where $X_{i,j}^v = \{x_{i,j,l}^v : l \in [1, m_{i,j}^v]\}$ and $Y_{i,j}^v = \{y_{i,j,r}^v : r \in [1, n_{i,j}^v]\}$; $G_{i,j}^v$ has a set-ordered graceful labelling $g_{i,j}$ such that $g_{i,j}(x_{i,j,1}^v) = 0$, $g_{i,j}(x_{i,j,l}^v) < g_{i,j}(x_{i,j,l+1}^v)$ for $l \in [1, m_{i,j}^v - 1]$, $g_{i,j}(x_{i,j,m_{i,j}^v}^v) < g_{i,j}(y_{i,j,1}^v)$, $g_{i,j}(y_{i,j,r}^v) < g_{i,j}(y_{i,j,r+1}^v)$ for $r \in [1, n_{i,j}^v - 1]$, $g_{i,j}(y_{i,j,n_{i,j}^v}^v) = q_{i,j}^v$. Let $N_v = \sum_{k=1}^t N_k$, where $N_i = \sum_{k=1}^{n_i} q_{i,k}^v$ for $i \in [1, t]$.

It is allowed that some $m_i = 0$, or $n_i = 0$, or $m_{i,j}^u = 0$, or $n_{i,j}^u = 0$, or $m_{i,j}^v = 0$, or $n_{i,j}^v = 0$ in the following discussion. We define a new labelling h and construct the desired graph H^* thorough the following steps A , B and C .

A . Labelling the edges of G and $G_{i,j}^u$ for $j \in [1, m_i]$ and $i \in [1, s]$, and the edges of $G_{k,r}^v$ for $r \in [1, n_k]$ and $k \in [1, t]$.

$A1$. We label the edges $x_{1,j,l}^u y_{1,j,r}^u$ of $G_{1,j}^u$ as: $h(x_{1,j,l}^u y_{1,j,r}^u) = f_{1,j}(x_{1,j,l}^u y_{1,j,r}^u)$ for edges $x_{1,j,l}^u y_{1,j,r}^u \in E(G_{1,j}^u)$, and we define the labels of edges $x_{i,j,l}^u y_{i,j,r}^u \in E(G_{i,j}^u)$ with $j \in [2, m_1]$ as

$$h(x_{1,j,l}^u y_{1,j,r}^u) = f_{1,j}(x_{1,j,l}^u y_{1,j,r}^u) + \sum_{k=1}^{j-1} q_{1,k}^u. \quad (1)$$

When $i \in [2, s]$, for edges $x_{i,1,l}^u y_{i,1,r}^u \in E(G_{i,1}^u)$ we define

$$h(x_{i,1,l}^u y_{i,1,r}^u) = f_{i,1}(x_{i,1,l}^u y_{i,1,r}^u) + \sum_{k=1}^{i-1} M_k, \quad (2)$$

and for edges $x_{i,j,l}^u y_{i,j,r}^u \in E(G_{i,j}^u)$, $j \in [2, m_i]$, we set

$$h(x_{i,j,l}^u y_{i,j,r}^u) = f_{i,j}(x_{i,j,l}^u y_{i,j,r}^u) + \sum_{k=1}^{i-1} M_k + \sum_{k=1}^{j-1} q_{i,k}^u. \quad (3)$$

$A2$. Labelling the edges of $G_{i,j}^v$ for $j \in [1, n_i]$ and $i \in [1, t]$. We label the edges $x_{t,n_t,l}^v y_{t,n_t,r}^v$ of G_{t,n_t}^v by $h(x_{t,n_t,l}^v y_{t,n_t,r}^v) = g_{t,n_t}(x_{t,n_t,l}^v y_{t,n_t,r}^v) + M_u$; and for edges $x_{t,j,l}^v y_{t,j,r}^v \in E(G_{t,j}^v)$, $j \in [1, n_t - 1]$,

we set

$$h(x_{t,j,l}^v y_{t,j,r}^v) = g_{t,j}(x_{t,j,l}^v y_{t,j,r}^v) + M_u + \sum_{k=j+1}^{n_t} q_{t,k}^v. \quad (4)$$

For $i \in [1, t-1]$, we define

$$h(x_{i,n_i,l}^v y_{i,n_i,r}^v) = g_{i,n_i}(x_{i,n_i,l}^v y_{i,n_i,r}^v) + M_u + \sum_{k=i+1}^t N_k \quad (5)$$

for edges $x_{i,n_i,l}^v y_{i,n_i,r}^v \in E(G_{i,1}^v)$, and set

$$h(x_{i,j,l}^v y_{i,j,r}^v) = g_{i,j}(x_{i,j,l}^v y_{i,j,r}^v) + M_u + \sum_{k=i+1}^t N_k + \sum_{k=j+1}^{n_i} q_{i,k}^u \quad (6)$$

for edges $x_{i,j,l}^v y_{i,j,r}^v \in E(G_{i,j}^v)$, $j \in [1, n_i - 1]$.

A3. We label the edges $u_i v_j$ of G as: $h(u_i v_j) = f(u_i v_j) + M_u + N_v$ for $u_i v_j \in E(G)$.

B. Labelling the vertices of G and $G_{i,j}^u$ for $j \in [1, m_i]$ and $i \in [1, s]$, and the vertices of $G_{k,r}^v$ for $r \in [1, n_k]$ and $k \in [1, t]$.

B1. We label the vertices of G as: $h(u_i) = f(u_i)$ for $u_i \in X$, and $h(v_j) = f(v_j) + M_u + N_v$ for $v_j \in Y$.

B2. We label the vertices of $G_{i,j}^u$ in the following way:

(1) $h(x_{1,j,1}^u) = h(u_1) = 0$ for $j \in [1, m_1]$; $h(x_{1,j,l}^u) = f_{1,j}(x_{1,j,l}^u)$ for $l \in [2, m_{1,1}^u]$ and $j \in [1, m_1]$; $h(y_{1,1,r}^u) = f_{1,1}(y_{1,1,r}^u)$ for $r \in [1, n_{1,1}^u]$; and $h(y_{1,j,r}^u) = f_{1,j}(y_{1,j,r}^u) + \sum_{k=1}^{j-1} q_{1,k}^u$ for $r \in [1, n_{1,1}^u]$ and $j \in [2, m_1]$. Under the vertex labels we verify the edge labels

$$\begin{aligned} h(x_{1,1,l}^u y_{1,1,r}^u) &= |h(x_{1,1,l}^u) - h(y_{1,1,r}^u)| = h(y_{1,1,r}^u) - h(x_{1,1,l}^u) \\ &= f_{1,1}(y_{1,1,r}^u) - f_{1,1}(x_{1,1,l}^u) = f_{1,1}(x_{1,1,l}^u y_{1,1,r}^u) \end{aligned}$$

for edges $x_{1,1,l}^u y_{1,1,r}^u \in E(G_{1,1}^u)$; and we compute the labels of edges $x_{1,j,l}^u y_{1,j,r}^u \in E(G_{1,j}^u)$ for $j \in [2, m_1]$ in the following

$$\begin{aligned} h(x_{1,j,l}^u y_{1,j,r}^u) &= |h(x_{1,j,l}^u) - h(y_{1,j,r}^u)| = h(y_{1,j,r}^u) - h(x_{1,j,l}^u) \\ &= f_{1,j}(y_{1,j,r}^u) + \sum_{k=1}^{j-1} q_{1,k}^u - f_{1,j}(x_{1,j,l}^u) = f_{1,j}(x_{1,j,l}^u y_{1,j,r}^u) + \sum_{k=1}^{j-1} q_{1,k}^u, \end{aligned}$$

which is equal to (1).

(2) For $i \in [2, s]$, we define $h(x_{i,j,1}^u) = f_{i,j}(x_{i,j,1}^u) + h(u_i) = h(u_i)$ and $h(x_{i,j,l}^u) = f_{i,j}(x_{i,j,l}^u) + h(u_i)$ for $l \in [2, m_{i,j}^u]$ and $j \in [1, m_i]$; $h(y_{i,1,r}^u) = h(u_i) + f_{i,1}(y_{i,1,r}^u) + \sum_{k=1}^{i-1} M_k$ for $r \in [1, n_{i,1}^u]$; and $h(y_{i,j,r}^u) = f_{i,j}(y_{i,j,r}^u) + \sum_{k=1}^{j-1} q_{i,k}^u + \sum_{k=1}^{i-1} M_k$ for $r \in [2, n_{i,j}^u]$ and $j \in [2, m_i]$. As verification of the edge labels, we can evaluate

$$h(x_{i,1,l}^u y_{i,1,r}^u) = |h(x_{i,1,l}^u) - h(y_{i,1,r}^u)| = h(y_{i,1,r}^u) - h(x_{i,1,l}^u) = f_{i,1}(x_{i,1,l}^u y_{i,1,r}^u) + \sum_{k=1}^{i-1} M_k \quad (7)$$

for edges $x_{i,1,l}^u y_{i,1,r}^u \in E(G_{i,1}^u)$; and for edges $x_{i,j,l}^u y_{i,j,r}^u \in E(G_{i,j}^u)$, $j \in [2, m_i]$,

$$\begin{aligned} h(x_{i,j,l}^u y_{i,j,r}^u) &= |h(x_{i,j,l}^u) - h(y_{i,j,r}^u)| = h(y_{i,j,r}^u) - h(x_{i,j,l}^u) \\ &= f_{i,j}(y_{i,j,r}^u) + \sum_{k=1}^{i-1} M_k + \sum_{k=1}^{j-1} q_{1,k}^u - f_{i,j}(x_{i,j,l}^u) \\ &= f_{i,j}(x_{i,j,l}^u y_{i,j,r}^u) + \sum_{k=1}^{i-1} M_k + \sum_{k=1}^{j-1} q_{1,k}^u. \end{aligned} \quad (8)$$

The above (7) and (8) are equal to (2) and (3), respectively.

B3. We label the vertices of $G_{i,j}^v$ in the following: (1') $h(x_{t,j,1}^v) = h(v_t) - g_{t,j}(x_{t,j,1}^v) = h(v_t)$ for $j \in [1, n_t]$; $h(x_{t,j,l}^v) = h(v_t) - g_{t,j}(x_{t,j,l}^v)$ for $l \in [2, n_{t,j}^v]$ and $j \in [1, n_t]$; $h(y_{t,n_t,r}^v) = h(v_t) - g_{t,n_t}(y_{t,n_t,r}^v) - M_u$ for $r \in [1, n_{t,n_t}^v]$; and

$$\begin{aligned} h(y_{t,j,r}^v) &= h(v_t) - g_{t,j}(y_{t,j,r}^v) - M_u - \sum_{k=j+1}^{n_t} q_{t,k}^v \\ &= f(v_t) - g_{t,j}(y_{t,j,r}^v) + N_v - \sum_{k=j+1}^{n_t} q_{t,k}^v \geq f(v_t) + \sum_{k=1}^{t-1} N_k \end{aligned}$$

for $r \in [1, n_{t,j}^v]$ and $j \in [1, n_t - 1]$. We verify the edge labels as follows:

$$\begin{aligned} h(x_{t,n_t,l}^v y_{t,n_t,r}^v) &= |h(x_{t,n_t,l}^v) - h(y_{t,n_t,r}^v)| = h(x_{t,n_t,l}^v) - h(y_{t,n_t,r}^v) \\ &= g_{t,n_t}(y_{t,n_t,r}^v) + M_u - g_{t,n_t}(x_{t,n_t,l}^v) = g_{t,n_t}(x_{t,n_t,l}^v y_{t,n_t,r}^v) + M_u \end{aligned}$$

for edges $x_{t,n_t,l}^v y_{t,n_t,r}^v \in E(G_{t,n_t}^v)$; and for edges $x_{t,j,l}^v y_{t,j,r}^v \in E(G_{t,j}^v)$ with $j \in [1, n_t - 1]$, we have

$$\begin{aligned} h(x_{t,j,l}^v y_{t,j,r}^v) &= h(x_{t,j,l}^v) - h(y_{t,j,r}^v) = g_{t,j}(y_{t,j,r}^v) + M_u + \sum_{k=j+1}^{n_t} q_{t,k}^v - g_{t,j}(x_{t,j,l}^v) \\ &= g_{t,j}(x_{t,j,l}^v y_{t,j,r}^v) + M_u + \sum_{k=j+1}^{n_t} q_{t,k}^v. \end{aligned}$$

The above edge labels are coincided with (4).

(2') For $i \in [1, t - 1]$, we set $h(x_{i,j,1}^v) = h(v_i) - g_{i,j}(x_{i,j,1}^v) = h(v_i)$, $h(x_{i,j,l}^v) = h(v_i) - g_{i,j}(x_{i,j,l}^v)$ for $l \in [2, n_{i,j}^v]$ and $j \in [1, n_i]$; $h(y_{i,n_i,r}^v) = h(v_i) - g_{i,n_i}(y_{i,n_i,r}^v) - M_u - \sum_{k=i+1}^t N_k$ for $r \in [1, n_{i,n_i}^v]$; and

$$\begin{aligned} h(y_{i,j,r}^v) &= h(v_i) - g_{i,j}(y_{i,j,r}^v) - M_u - \sum_{k=i+1}^t N_k - \sum_{k=j+1}^{n_i} q_{i,k}^v \\ &= f(v_i) - g_{i,j}(y_{i,j,r}^v) + \sum_{k=1}^i N_k - \sum_{k=j+1}^{n_i} q_{i,k}^v \geq f(v_i) + \sum_{k=1}^{i-1} N_k \end{aligned}$$

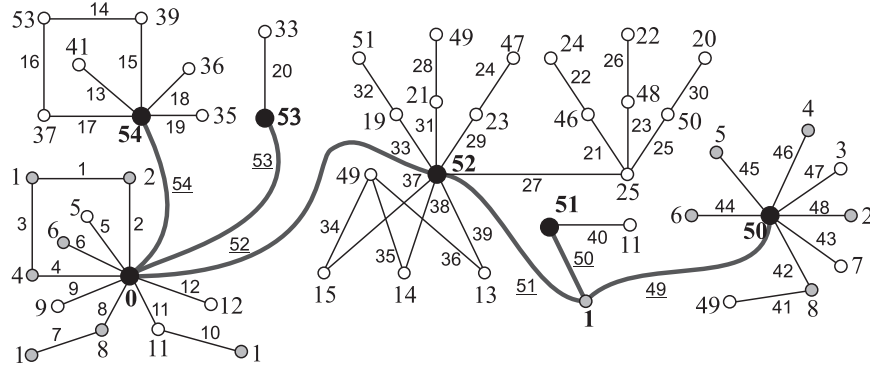


Figure 3: The graph H^* obtained in Theorem 4 has an in-improper graceful labelling.

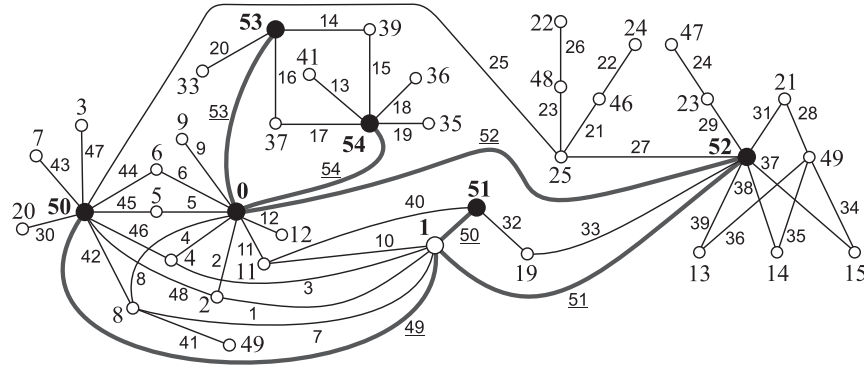


Figure 4: A graceful graph obtained from H^* shown in Figure 3 by identifying those vertices having the same labels into one vertex.

Problem. Determine $in-(k_{in}^{\max}, l_{in}^{\max})$ -imgls, $in-(k_{in}^{\min}, l_{in}^{\min})$ -imgls and $out-(k_{out}^{\min}, l_{out}^{\min})$ -imgls of trees.

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